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Chapter I

1. Introduction

Lambert problem of space researches is concerned with the determination of an orbit from two position vectors and the time of flight(Danby 1988). It has very important applications in the areas of rendezvous, targeting, guidance (Noton 1998) and interplanetary missions(Eagle 1991).

Solutions to Lambert's problem abound in the literature, as they did even in Lambert's time shortly after his original formulation in 1716. Examples are Lambert's original geometric formulation, which provides equations to determine the minimum-energy orbit, and the original Gaussian formulation, which gives geometrical insight into the problem.

Up to the year 1965, a fairly comprehensive list of references on Lambert's problem are given in references (Escobal 1965), (Herrick 1971) and(Battin 1964). (Lancaster and Blanchard 1969) also(Mansfield 1989) established unified forms of Lambert's problem, (Gooding 1990) developed a procedure for the solution, and in (1995), (Thorne and Bain 1995) developed a direct solution using series inversion technique.Recently (Sharaf 2003) developed an algorithm for the universal Lambert's problem based on iterative scheme that could be made convrage for all coin motion.

Each of the above methods is characterized primarily by: (1) a particular form of the time of flight equation and, (2) a particular independent variable to be used in an iteration algorithm to determine the orbital elements.

One of the most compact and computational efficient form of Lambert's problem is that of Battin (cited in reference(Bond and Allman,1996). In this form, the time of flight equation is universal (i.e., includes elliptic, parabolic, and hyperbolic orbits) as a well-behaved function of a single, physically significant independent variable.

The present thesis is devoted for the study of the boundary value problem in its universal form,and it comprise two parts

In the first part, the properties of the orbital boundary value problem are presented including terminal velocity vectors with different coordinates and the minimum energy orbit with it's various orbital elements. The fundamental ellipse is discussed, together with the various forms of its parameters. All of these properties are proved mathematically and illustrated geometrically.

The second part of the thesis is devoted to the solution of Lambert problem for different conic sections .In this respect we considered :

- "Gauss Method": for elliptic orbits, the equations of the method together with the its computational algorithm are presented .
- "The iterative method" : for elliptic orbits ,by which the values of semi major axis and each Lagrange coefficients "f" and "g", are computed so as to determine the initial velocity v_1.

Also some methods for solving universal Lambert problem are discussed, including:

- "Linear terminal velocity constrain": for which, the basic equations, computational algorithms and some numerical applications are given.

- "Computational algorithms" to solve universal Lambert problem, and,the basic equations, some numerical applications are given.
- "Battin's method": for which the basic equations and computational algorithms are given in full details .In addition, we implement the method to compute the geometric characteristics of the boundary value problem (demonstrated in the first part). Finally we made use of these computedgeometric characteristics as criteria for accuracy checks of the calculations. The algorithms is applied to 14 orbits of different eccentricity ,the numerical results are extremely accurate.

Chapter II

Basic Topics

In this chapter, some topics will be given due to their important roles in the analysis of the subsequent chapters

2-1Basic topicsand number theory

2-1-1 Continued fraction

In fact, continued fraction expansions are, generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series. Due to the importance of accurate evaluations of the space orbital maneuvers and the efficiency of continued fractions, we purpose to use them as the computational tools for evaluating the included functions .

2-1-2 Top- Down Continued Fraction Evaluation

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the nth convergent were accumulated separately with three-term recurrence formulae. The draw back of the first method is, obviously, having to decide far down the fraction to being in order to ensure convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint .

Gautschi (1967) proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as:

$$c = \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \ddots}}}$$

then initialize the following parameters

$$a_1 = 1,$$
$$b_1 = n_1/d_1,$$
$$c_1 = n_1/d_1$$

and iterate (k=1,2,…) according to :

$$a_{k+1} = \frac{1}{1 + \left[\frac{n_{k+1}}{d_k d_{k+1}}\right] a_k}$$

$$b_{k+1} = [a_{k+1} - 1] b_k,$$

$$c_{k+1} = c_k + b_{k+1}.$$

In the limit, the c sequence converges to the value of the continued fraction.

Continued fraction method was used in many problems in astrophysics (e.g. Sharaf, 2006,Sharaf et.al 2004) as well as in special functions of astrodynamics (e.g.Sharaf and Banajh 2001,Sharaf,and Najmuldeen,2001).

2-2Basic topics from space dynamics

2-2-1 Initial value problems

The initial value problem is: Given initial conditions $\mathbf{r}_0 = \mathbf{r}(t_0)$ and $\mathbf{v}_0 = \mathbf{v}(t_0)$ for the position and velocity vectors at time t_0,and given a second time t ,find $\mathbf{r}(t)$ and $\mathbf{v}(t)$

2-2-2 Basic relations between position and time

The basic relations between position and time for the different conic sections as:

$$M = E - e\sin E \quad ; \quad e \prec 1 \quad \text{For elliptic orbits}$$

$$M = \tan^3 \frac{1}{2} f + 3\tan \frac{1}{2} f \quad ; \quad e = 1 \quad \text{For parabolic orbits}$$

$$M = e\sinh H - H \quad ; \quad e \succ 1 \text{For hyperbolic orbits}$$

The first equation is known as Keplar's equation, the second equation as Barker's equation, while the third equation is the hyperbolic form of Kepler's equation. The angle f is the true anomaly ,E and H are respectively ,the elliptic eccentric anomaly and the hyperbolic eccentric anomaly. The mean anomaly M is related to the time t for the respect orbits by:

$$M = \sqrt{\frac{\mu}{a^3}}(t - \tau),$$

$$M = 6\sqrt{\frac{\mu}{p^3}}(t - \tau) \;,$$

$$M = \sqrt{\frac{\mu}{(-a)^3}}(t - \tau),$$

Where a, τ is time are respectively the semi-major axis of the orbit and the time of pericentre passeg , where p is, the semi-latus rectum of the orbit(or simply the parameter).

2-2-3 Two – body formulations

The equation describing the relative motion of the two bodies of masses m_1 and m_2 in rectangular coordinates is :

$$\frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\,\mathbf{r} \,, \qquad (2\text{-}1)$$

where μ is the gravitational parameter (universal gravitational constant times the sum of the two masses) $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity vectors respectively ,given in components as :

$$\mathbf{r} = x\ \mathbf{i}_x + y\ \mathbf{i}_y + z\ \mathbf{i}_z \,,$$

$$\mathbf{v} = \dot{x}\ \mathbf{i}_x + \dot{y}\ \mathbf{i}_y + \dot{z}\ \mathbf{i}_z \,,$$

$\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are the unit vectors along the coordinate axes x, y and z respectively and

$r = \left(x^2 + y^2 + z^2\right)^{1/2}$.

Equation (2-1) is unchanged if we replace $\mathbf{r}$ with $-\mathbf{r}$. Thus Equation (2-1) gives the motion of the body of mass m_2 relative to the body of the mass m_1, or the motion of m_1 relative to m_2. Also if we replace t with – t, Equation (2-1) is unchanged.

At any time, $\mathbf{r}$ and $\mathbf{v}$ can be expressed as:

$$\mathbf{r} = L\,\mathbf{i}_e + T\,\mathbf{i}_p \,, \qquad (2\text{-}2)$$

$$\mathbf{v} = \dot{L}\,\mathbf{i}_e + \dot{T}\,\mathbf{i}_p \quad , \tag{2-3}$$

where (L, T) are the pericentre coordinates of one of the bodies in its orbit about the other body and $(\dot{L}, \dot{T})$ are their time derivatives. These coordinates are of different forms(Battin 1999) for the different types (elliptic, parabolic, hyperbolic) of the two body motion and are not needed to be specified here. The unit vectors $\mathbf{i}_e$, $\mathbf{i}_p$ and $\mathbf{i}_h$ are selected such that, $\mathbf{i}_e$ and $\mathbf{i}_p$ in the body's own orbital plane with $\mathbf{i}_e$ in the direction of pericentre, while $\mathbf{i}_p$ and $\mathbf{i}_h$ are chosen to make the coordinate system right-handed.

Among the integrals of the two-body problem are the conservation of angular momentum vector **h** where,

$$\mathbf{h} = \sqrt{\mu\,p}\;\mathbf{i}_h = \sqrt{\mu\,p}\;(\mathbf{i}_e \times \mathbf{i}_p) = \mathbf{r} \times \mathbf{v} \tag{2-4}$$

and the energy integral

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{2-5}$$

From Equations (2-2), (2-3) and (2-4) we get :

$$L\dot{T} - \dot{L}T = \sqrt{\mu\,p}\,. \tag{2-6}$$

2-2-4 Lagrange's fundamental invariants

The basic equations governing the relative motion of two bodies are nonlinear [see Equation (2-1)] so that , priori ,we should not expect closed form expressions for the position and velocity vectors $\mathbf{r}$ and $\mathbf{v}$ to exist as time dependent quantities. Under any circumstances, though, power series developments may be obtained. Indeed, the coefficients in Taylor series expansion :

$$\mathbf{r}(t) = \mathbf{r}_0 + (t - t_0)\left.\frac{d\mathbf{r}}{dt}\right|_0 + \frac{1}{2!}(t - t_0)^2 \left.\frac{d^2\mathbf{r}}{dt^2}\right|_0 + \frac{1}{3!}(t - t_0)^3 \left.\frac{d^3\mathbf{r}}{dt^3}\right|_0 + ...$$

can be found from the Equation of motion (2-1) and its higher derivatives.

Successive differentiation of Equation (2-1) involves higher derivatives of the quantity μ / r^3 ,a calculation that fortunately, can be expedited in a convenient and quite interesting manner. For, if we define :

$$\varepsilon = \mu / r^3$$

Then

$$.\frac{d\varepsilon}{dt} = -3\frac{\mu}{r^4}\frac{dr}{dt} = -3\varepsilon\frac{1}{r}\frac{dr}{dt}$$

Now define

$$\lambda = \frac{1}{r}\frac{dr}{dt} = \dot{r} / r \text{ .}$$

Since

$$< \mathbf{r}, \mathbf{v} > = r\dot{r},$$

then λ could be written as:

$$\lambda = \frac{\dot{r}}{r} = \frac{1}{r^2} < \mathbf{r}, \mathbf{v} > .$$

From Equation (2-1) we get:

$$< \mathbf{r}, \ddot{\mathbf{r}} > \; = -\frac{\mu}{r^3} < \mathbf{r}, \mathbf{r} > \; = -\frac{\mu}{r^3} r^2 = -\frac{\mu}{r},$$

since

$$< \mathbf{r}, \mathbf{v} > \frac{\dot{r}}{r^3} = \frac{\dot{r}^2}{r^2} = \lambda^2,$$

then

$$\frac{d\lambda}{dt} = \frac{v^2}{r^2} - \frac{\mu}{r^3} - 2\lambda^2 = \frac{v^2}{r^2} - \varepsilon - 2\lambda^2 .$$

Finally ,we define

$$\Psi = \frac{v^2}{r^2} = \frac{1}{r^2} < \mathbf{v}, \mathbf{v} >,$$

so

$$\frac{d\Psi}{dt} = \frac{2}{r^2} < v, \dot{v} > - \frac{2v^2}{r^3} \dot{r} = \frac{2}{r^2} < \dot{r}, \ddot{r} > - \frac{2v^2}{r^3} \dot{r} .$$

From Equation (2-1) we have:

$$< \dot{\mathbf{r}}, \ddot{\mathbf{r}} > = -\frac{\mu}{r^3} < \mathbf{r}, \dot{\mathbf{r}} > = -\frac{\mu}{r} \frac{1}{r^2} < \mathbf{r}, \mathbf{v} > = -\frac{\mu}{r} \lambda ,$$

also

$$\frac{v^2}{r^3} \dot{r} = \frac{v^2}{r^2} \frac{\dot{r}}{r} = \Psi \lambda$$

then

$$\frac{d\Psi}{dt} = -\frac{2\mu\lambda}{r^3} - 2\Psi\lambda = -2\varepsilon\lambda - 2\Psi\lambda = -2\lambda(\varepsilon + \Psi)$$

The term *fundamental invariants* has been used for $\varepsilon, \lambda, \Psi$ - they are “invariants” because they are independent of the selected coordinate system and "fundamental" because they form a closed set under the operation of time differentiation .Thus ,to calculate the various derivatives of the position vector $\mathbf{r}$, we have successively differentiate :

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad ; \quad \frac{d\mathbf{v}}{dt} = -\varepsilon\,\mathbf{r}$$

using the relations:

$$\frac{d\varepsilon}{dt} = -3\varepsilon\lambda; \quad \frac{d\lambda}{dt} = \Psi - \varepsilon - 2\lambda^2; \quad \frac{d\Psi}{dt} = -2\lambda(\varepsilon + \Psi)$$

where the quantities $\varepsilon, \lambda, \Psi$ are defined as:

$$\varepsilon = \mu / r^3 \quad ; \quad \lambda = \frac{1}{r^2} < \mathbf{r}, \mathbf{v} > \quad ; \quad \Psi = \frac{1}{r^2} < \mathbf{v}, \mathbf{v} > .$$

In this manner ,we obtain:

$$\dot{\mathbf{r}} = \mathbf{v}$$

$$\ddot{\mathbf{r}} = -\varepsilon\,\mathbf{r}$$

$$\dddot{\mathbf{r}} = 3\varepsilon\lambda - \varepsilon\,\mathbf{v}$$

$$, \mathbf{r}^{iv} = \left(-15\varepsilon\lambda^2 + 3\varepsilon\Psi - 2\varepsilon^2\right)\mathbf{r} + 6\varepsilon\lambda\,\mathbf{v}$$

indicating that the position vector **r** at any time t can be represented in terms of the

position and velocity vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ at time t_0 in the form :

$$\mathbf{r}(t) = F(t)\mathbf{r_0} + G(t)\dot{\mathbf{r}}_\mathbf{0} \qquad (2\text{-}7\text{-}1)$$

$$\dot{\mathbf{r}}\ (t) = F_t(t)\mathbf{r}_0 + G_t(t)\dot{\mathbf{r}}_\mathbf{0} \qquad (2\text{-}7\text{-}2)$$

2-2-5 Universal formulations for conic orbits

Importance of the universal formulations

During space mission all types of the two body motion (elliptic,parabolic,or hyperbolic) appear. For examples the escape from the departure planet and the capture by the target planet involve hyperbolic orbits, while the intermediate stage of the mission commonly depicted as a heliocentric ellipse ,may also be heliocentric parabola or hyperbola. In addition, in some systems, the type of an orbit is occasionally changed by perturbing forces during finite interval of time. Thus far we have been obliged to use different functional representations for motion depending upon the energy state (elliptic, parabolic, or hyperbolic) and a simulation code must then contain branching to handle a switch from one state to another .In cases where this switching is not smooth, branching can occur many times during a single integration time-step causing some numerical “chatter”. Consequently ,universal formulations are desperately needed so that ,orbit predictions will be free of the troubles ,since a single functional representation suffices to describe all possible states.

Formulations

It is convenient to write α for the reciprocal of the semi-major axis, so that :

$$\alpha \equiv \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu} \ ,$$

where $r = |\mathbf{r}|$ and $v = |\dot{\mathbf{r}}|$ .Depending on the sign of α ,or the value of the eccentricity e,the type of the orbit is determined such as:

$$\alpha = \begin{cases} \succ 0 \ (\text{or } e \prec 1) & \text{for elliptic orbits,} \\ = 0 \ (\text{ or } e = 1) & \text{for parabolic orbits,} \\ \prec 0 \ (\text{ or } e \succ 1) & \text{for hyperbolic orbits.} \end{cases}$$

Different formulations for various two body orbits can be unified by using :

A-time transformation formula ,

B- new family of transcendental functions.

Each of these points will be considered as follows.

A-Time transformation formula

Regarding this point ,we shall use Sundman's (Battin 1999) time transformation defined by:

$$\sqrt{\mu}\,\frac{dt}{d\chi} = r\,,$$

where χ is to be considered as a new independent variable –a kind of *generalized anomaly*. For the initial time t_0 and a second time t ,the variable χ can be related to the classical anomalies at these times by:

$$\chi = \begin{cases} \sqrt{a}\ \left(E - E_0\right) & \text{if } \alpha \succ 0, \\ \sqrt{p}\left(\tan\frac{1}{2}f - \tan\frac{1}{2}f_0\right) & \text{if } \alpha = 0, \\ \sqrt{-a}\ \left(H - H_0\right) & \text{if } \alpha \prec 0. \end{cases}$$

It could be shown that (Battin 1999) :

$$\frac{dr}{d\chi} = \sigma = \frac{1}{\sqrt{\mu}}\langle \mathbf{r}, \mathbf{v}\rangle, \tag{2-8}$$

$$\frac{d^2 r}{d\chi^2} = 1 - \alpha\, r = \frac{d\sigma}{d\chi}, \tag{2-9}$$

$$\frac{d\mathbf{r}}{d\chi} = \frac{r}{\sqrt{\mu}}\mathbf{v}, \tag{2-10}$$

$$\frac{d^2\mathbf{r}}{d\chi^2} = \frac{\sigma}{\sqrt{\mu}}\mathbf{v} - \frac{1}{r}\mathbf{r}, \tag{2-11}$$

$$\frac{d^2\sigma}{d\chi^2} + \alpha\,\sigma = 0, \tag{2-12}$$

$$\frac{d^3 r}{d\chi^3} + \alpha\frac{dr}{d\chi} = 0, \tag{2-13}$$

$$\frac{d^4 t}{d\chi^4} + \alpha\frac{d^2 t}{d\chi^2} = 0, \tag{2-14}$$

$$\frac{d^3\mathbf{r}}{d\chi^3} + \alpha\frac{d\,\mathbf{r}}{d\chi} = \mathbf{0}. \tag{2-15}$$

B- The new family of transcendental functions

Regarding the second point mentioned above ,we shall consider for the family of transcendental functions, those defined by:

$$U_n(\chi;\alpha) = \chi^n \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha\,\chi^2)^k}{(n+2k)!}\ , \tag{2-16}$$

what concerns us in the subsequent analysis are the following relations satisfied by the U's functions:

$$U_n + \alpha\,U_{n+2} = \frac{\chi^n}{n!}\ , \tag{2-17}$$

$$\frac{d^{m+1}U_n}{d\chi^{m+1}} + \alpha \frac{d^{m-1}U_n}{d\chi^{m-1}} = 0; n = 0,1,2,\cdots, m, \tag{2-18}$$

$$U_0(0) = 1 \quad ; \quad U_n(0) = 0 \quad \forall n \geq 1 \quad . \tag{2-19}$$

The relations of the functions $U_j(\chi;\alpha)$ $j = 0,1,2,3$ to the elementary functions are given as:

$$U_0(\chi;\alpha) = \begin{cases} 1 & \text{if } \alpha = 0, \\ \cos\left(\sqrt{\alpha}\chi\right) & \text{if } \alpha \succ 0, \\ \cosh\left(\sqrt{-\alpha}\chi\right) & \text{if } \alpha \prec 0. \end{cases} \tag{2-20}$$

$$U_1(\chi;\alpha) = \begin{cases} \chi & \text{if } \alpha = 0, \\ \sin\left(\sqrt{\alpha}\chi\right)/\sqrt{\alpha} & \text{if } \alpha \succ 0, \\ \sinh\left(\sqrt{-\alpha}\chi\right)/\sqrt{-\alpha} & \text{if } \alpha \prec 0. \end{cases} \tag{2-21}$$

$$U_2(\chi;\alpha) = \begin{cases} \frac{1}{2}\chi^2 & \text{if } \alpha = 0, \\ \left[1 - \cos\left(\sqrt{\alpha}\chi\right)\right] / \alpha & \text{if } \alpha \succ 0, \\ \left[\cosh\left(\sqrt{-\alpha}\chi\right) - 1\right] / (-\alpha) & \text{if } \alpha \prec 0. \end{cases} \tag{2-22}$$

$$U_3(\chi\ ;\alpha)=\begin{cases}\frac{1}{6}\chi^3 & \text{if } \alpha=0,\\ \left[\sqrt{\alpha}\chi-\sin\left(\sqrt{\alpha}\chi\right)\right]/\alpha\sqrt{\alpha} & \text{if } \alpha\succ 0,\\ \left[\sinh\left(\sqrt{-\alpha}\chi\right)-\sqrt{-\alpha}\chi\right]/(-\alpha\sqrt{-\alpha}) & \text{if } \alpha\prec 0.\end{cases}\quad (2\text{-}23)$$

The functions $U_0, U_1, \cdots, U_n$ are linearly independent. Finally we have:

$$\frac{dU_n}{d\chi}=U_{n-1}\quad;\quad n=1,2,\cdots \quad (2\text{-}24)$$

$$\frac{dU_0}{d\chi}=-\alpha\, U_1. \quad (2\text{-}25)$$

2-2-6 Orbital parameters in terms of the U's **functions**

I- σ **in terms of the** U's **functions**

Let $m=1$ in Equation (2-15) we get :

$$\frac{d^2U_n}{d\chi^2}+\alpha U_n=0\ ;\ n=0,1\ . \quad (2\text{-}26)$$

From this equation ,Equation (2-12) and the fact act that U_0 and U_1 are linearly independent we get:

$$\sigma=A_0U_0+A_1U_1, \quad (2\text{-}27)$$

where A's are constants.

From Equations (2-9),(2-19) and (2-27) we get at $\chi=0$,

$$A_0=\sigma_0\ . \quad (2\text{-}28\text{-}1)$$

From Equations (2-19) and(2-27) we get on using Equations (2-24) and (2-25) that:

$$1-\alpha\, r = -A_0\, \alpha\, U_1 + A_1 U_0 \xrightarrow{\text{at } \chi=0}$$

$$A_1 = 1-\alpha\, r_0 . \tag{2-28-2}$$

From Equations (2-28) ,Equation (2-27) becomes :

$$\sigma = \sigma_0 U_0 + \left(1-\alpha\, r_0\right) U_1 \ , \tag{2-29}$$

which is the required equation of σ in terms of the U's functions

II- r **in terms of the** U's **functions**

Let $m = 2$ in Equation (2-18) we get :

$$\frac{d^3 U_n}{d\chi^3} + \alpha \frac{dU_n}{d\chi} = 0 \ ; \ n = 0,1,2. \tag{2-30}$$

From this equation ,Equation (2-13) and the fact act that U_0 , U_1 and U_2 are linearly independent

we get:

$$r = B_0 U_0 + B_1 U_1 + B_2 U_2 \, , \tag{2-31}$$

where B's are constants.

From Equations (2-19) and (2-31) we get at $\chi = 0$,

$$B_0 = r_0 \ . \tag{2-32-1}$$

From Equations (2-8) and (2-31) we get on using Equations (2-24) and (2-25) that:

$$\sigma = -r_0\, \alpha\, U_1 + B_1 U_0 + B_2 U_1 \xrightarrow{\text{at } \chi=0}$$

$$B_1 = \sigma_0 . \tag{2-32-2}$$

From Equations (2-9) and (2-31) we get on using Equations (2-24) and (2-25) that :

$$1-\alpha\, r=-r_0\,\alpha\, U_0-\alpha\, B_1 U_1+B_2 U_0 \xrightarrow{\text{at }\chi=0}$$

$1-\alpha\, r_0=-\alpha\, r_0+B_2 \Rightarrow$

$$B_2=1. \tag{2-32-3}$$

From Equations (2-32) ,Equation (2-31) becomes :

$$r=r_0 U_0+\sigma_0 U_1+U_2, \tag{2-33}$$

which is the required equation of rin terms of the U's functions

III-Universal Kepler's equation

Let $m=3$ in Equation (2-18) we get :

$$\frac{d^4 U_n}{d\chi^4}+\alpha\frac{d^2 U_n}{d\chi^2}=0\ ;\ n=0,1,2,3\ .$$

From this equation ,Equation (2-14) and the fact act that U_0 , U_1 , U_2 and U_3 are linearly

independent we can write :

$$\sqrt{\mu}\left(t-t_0\right)=\gamma_0 U_0+\gamma_1 U_1+\gamma_2 U_2+\gamma_3 U_3, \tag{2-34}$$

where γ 's are constants.

From Equations (2-19) and (2-34) we get at $\chi=0$,or $t=t_0$

$$\gamma_0=0\ . \tag{2-35-1}$$

From Sundman's and Equation (2-34) we get on using Equations (2-24) and (2-25) that:

$$\sqrt{\mu}\frac{dt}{d\chi}=r=-\gamma_0\,\alpha\, U_1+\gamma_1 U_0+\gamma_2 U_1+\gamma_3 U_2 \xrightarrow{\text{at }\chi=0}$$

$$\gamma_1=r_0. \tag{2-35-2}$$

From Equations (2-8) and 2-34) we get on using Equations (2-24) and (2-25) that :

$$\sqrt{\mu}\,\frac{d^2t}{d\chi^2}=\frac{dr}{d\chi}=\sigma=-\gamma_0\,\alpha\,U_0-\gamma_1\,\alpha\,U_1+\gamma_2 U_0+\gamma_3 U_1\xrightarrow{\text{at }\chi=0}$$

$$\sigma_0=-\gamma_0\,\alpha+\gamma_2\Rightarrow$$

$$\gamma_2=\sigma_0\quad. \tag{2-35-3}$$

From Equations (2-9) and(2-34) we get on using Equations (2-24) and (2-25) that :

$$\frac{d\sigma}{d\chi}=1-\alpha\,r=\gamma_0\,\alpha^2\,U_1-\gamma_1\,\alpha\,U_0-\gamma_2\,\alpha\,U_1+\gamma_3 U_0\xrightarrow{\text{at }\chi=0}$$

$$1-\alpha\,r_0=-\gamma_1\,\alpha+\gamma_3\Rightarrow$$

$$\gamma_3=1\ . \tag{2-35-4}$$

From Equations (2-35) ,Equation (2-34) becomes :

$$\sqrt{\mu}\left(t-t_0\right)=r_0U_1+\sigma_0U_2+U_3\,. \tag{2-36}$$

This equation is the universal Kepler'sequation

IV-Lagrangian coefficients

From Equations (2-30) ,Equation (2-15) and the fact act that U_0 ,U_1 and U_2 are linearly independent we can write :

$$\mathbf{r}=\mathbf{a}_0U_0+\mathbf{a}_1U_1+\mathbf{a}_2U_2\,, \tag{2-37}$$

where **a'**s are vector constants.

From Equations (2-19) and (2-37) we get at $\chi=0$,

$$\mathbf{a}_0=\mathbf{r}_0\quad. \tag{2-38-1}$$

Differentiating Equation (2-37) and then using Equations (2-10) ,(2-24) and(2-25) we get :

$$\frac{r}{\sqrt{\mu}}\mathbf{v} = -\alpha\,U_1\mathbf{a}_0 + \mathbf{a}_1 U_0 + \mathbf{a}_2 U_1, \qquad (2\text{-}39)$$

from which we get:

$$\mathbf{a}_1 = \frac{r_0}{\sqrt{\mu}}\mathbf{v}_0 \;. \qquad (2\text{-}38\text{-}2)$$

Differentiating Equation (2-37) twice and then using Equation (2-11) we get :

$$\frac{\sigma}{\sqrt{\mu}}\mathbf{v} - \frac{1}{r}\mathbf{r} = -\alpha U_0\mathbf{a}_0 - \alpha U_1\mathbf{a}_1 + U_0\mathbf{a}_2 \;, \qquad (2\text{-}40)$$

from which we get:

$$\mathbf{a}_2 = \frac{\sigma_0}{\sqrt{\mu}}\mathbf{v}_0 - \frac{1}{r_0}\mathbf{r}_0 + \alpha\,\mathbf{r}_0 \;. \qquad (2\text{-}38\text{-}3)$$

Using Equations (2-38) into Equation (2-37) we get:

$$\mathbf{r} = \mathbf{r}_0 U_0 + \frac{r_0}{\sqrt{\mu}}\mathbf{v}_0 U_1 + U_2\left(\frac{\sigma_0}{\sqrt{\mu}}\mathbf{v}_0 - \frac{1}{r_0}\mathbf{r}_0 + \alpha\,\mathbf{r}_0\right),$$

which can be written as :

$$\mathbf{r} = F\,\mathbf{r}_0 + G\,\mathbf{v}_0,$$

where

$$F = U_0 + \left(\alpha - \frac{1}{r_0}\right)U_2 = U_0 + \alpha\,U_2 - \frac{1}{r_0}U_2 \xrightarrow{\text{Using Equation(2-17) with n=0}}$$

$$F = 1 - \frac{1}{r_0}U_2,$$

$$G = \frac{r_0}{\sqrt{\mu}}U_1 + \frac{\sigma_0}{\sqrt{\mu}}U_2 .$$

Now using Equations (2-38) into Equation (2-39) we get:

$$\mathbf{v} = \frac{\sqrt{\mu}}{r}\left\{-\alpha U_1 \mathbf{r}_0 + \frac{r_0 U_0}{\sqrt{\mu}}\mathbf{v}_0 + U_1\left(\frac{\sigma_0}{\sqrt{\mu}}\mathbf{v}_0 - \frac{1}{r_0}\mathbf{r}_0 + \alpha\,\mathbf{r}_0\right)\right\},$$

which can be written as:

$$\mathbf{v} = F_t \mathbf{r}_0 + G_t \mathbf{v}_0,$$

where

$$F_t = -\frac{\alpha\, U_1 \sqrt{\mu}}{r} - \frac{\sqrt{\mu}}{r r_0} U_1 + \frac{\alpha U_1 \sqrt{\mu}}{r} = -\frac{\sqrt{\mu}}{r r_0} U_1,$$

$$G_t = \frac{\sqrt{\mu}}{r}\left\{\frac{r_0 U_0}{\sqrt{\mu}} + \frac{U_1 \sigma_0}{\sqrt{\mu}}\right\} = \frac{1}{r}\{\, r_0 U_0 + U_1 \sigma_0\} \xrightarrow{\text{Using Equation(2-33)}}$$

$$G_t = 1 - \frac{1}{r} U_2.$$

Now, collecting the above equations we get for the universal initial value problem ,the

formulations:

$$\mathbf{r} = F\mathbf{r}_0 + G\,\mathbf{v}_0 \quad ; \quad \mathbf{v} = F_t \mathbf{r}_0 + G_t \mathbf{v}_0$$

$$F = 1 - \frac{1}{r_0} U_2, \quad ; \quad G = \frac{r_0}{\sqrt{\mu}} U_1 + \frac{\sigma_0}{\sqrt{\mu}} U_2,$$

$$F_t = -\frac{\sqrt{\mu}}{r r_0} U_1, \quad ; \quad G_t = 1 - \frac{1}{r} U_2,$$

$$r = r_0 U_0 + \sigma_0 U_1 + U_2 .$$